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薄板哈密顿求解体系及其变分原理

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摘　要：将哈密顿求解体系推广应用于薄板弯曲问题。首先导出薄板哈密顿对偶微分方程，然后导出薄板哈密顿变分原理的泛函表示式 P_H。有两点值得指出：第一，以挠度 w、转角 y_x、弯矩 M_x 和等效剪力 V_x 取为对偶变量，与相关文献的取法不同。第二，对于薄板问题，由 Hellinger-Reissner 泛函 P_{HR} 导出哈密顿泛函 P_H 时既要消元，又要增元，与在厚板问题中只需要消元的推导方法不同。薄板哈密顿求解体系的理论成果将为研究薄板解析解和有限元解提供新的有效工具。

关键词：哈密顿求解体系；薄板理论；对偶方程；变分原理；乘子法

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HAMILTONIAN SOLUTION SYSTEM FOR THIN PLATES AND ITS VARIATIONAL PRINCIPLE

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Abstract: The Hamiltonian solution system is generalized to the thin plate problems. Firstly, the Hamiltonian dual differential equations for thin plates are derived. Then, the functional expression of Hamiltonian variational principle, Π_H, is obtained. Differing from those proposed in related reference, the deflection w, the rotation y_x, the moment M_x and the equivalent shear force V_x are taken as the dual variables; For thin plate problems, both elimination and addition of variables are needed when the Hamiltonian functional Π_H is derived using the Hellinger-Reissner functional Π_{HR}. The process is different from that used in the thick plate problems, where only elimination is utilized. The theoretical achievements of the Hamiltonian system for thin plates provide new effective tools for analytical and finite element solutions of thin plates.

Key words: Hamiltonian solution system; thin plate theory; dual equation; variational principle; Lagrange multiplier

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1 引言

弹性力学哈密顿求解体系在[1,2]中有系统论述。文献[3,4]在各向同性材料情况下，将文献[1]中的正交关系分解为两个子正交关系。文献[5]将哈密顿求解体系推广到厚板弯曲问题，文献[6,7]将它推广到薄板弯曲问题，其中应用了薄板弯曲与平面弹性问题之间的比拟关系，并在此比拟关系基础上推导出哈密顿对偶微分方程和变分原理。

本文仍讨论薄板弯曲问题的哈密顿求解体系，但不采用文献[6,7]中所用的对偶变量和推导方法。首先，采用挠度 w，转角ψ_x，弯矩M_x和等效剪力V_x作为对偶变量建立哈密顿对偶微分方程。其次由 Hellinger-Reissner 薄板变分原理泛函Π_{HR}出发，推导出哈密顿薄板变分原理泛函Π_H，这里既要采用消元，又要采用换元乘子法[8]进行增元。

2 薄板哈密顿对偶方程

2.1 薄板方程的回顾[9,10]

几何方程——挠度 w 是独立的位移变量，由它可导出转角ψ_x,ψ_y以及曲率扭率$\kappa_x,\kappa_y,\kappa_{xy}$：

$$\psi_x=\frac{\partial w}{\partial x}\qquad \psi_y=\frac{\partial w}{\partial y} \tag{1}$$

$$\kappa_x=-\frac{\partial^2 w}{\partial x^2}\qquad \kappa_y=-\frac{\partial^2 w}{\partial y^2}\qquad \kappa_{xy}=-2\frac{\partial^2 w}{\partial x\partial y} \tag{2}$$

平衡微分方程为：

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}=Q_x \tag{3a}$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}=Q_y \tag{3b}$$

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q=0 \tag{3c}$$

其中弯矩扭矩M_x，M_y，M_{xy}是三个独立的内力分量，由它们可导出剪力Q_x和Q_y，q 是竖向荷载集度。

物理方程——可用式(4)或式(5)表示：

$$\begin{Bmatrix}M_x\\M_y\\M_{xy}\end{Bmatrix}=D\begin{bmatrix}1&\mu&0\\\mu&1&0\\0&0&\frac{1-\mu}{2}\end{bmatrix}\begin{Bmatrix}\kappa_x\\\kappa_y\\\kappa_{xy}\end{Bmatrix} \tag{4}$$

$$\begin{Bmatrix}\kappa_x\\\kappa_y\\\kappa_{xy}\end{Bmatrix}=\frac{1}{(1-\mu^2)D}\begin{bmatrix}1&-\mu&0\\-\mu&1&0\\0&0&2(1+\mu)\end{bmatrix}\begin{Bmatrix}M_x\\M_y\\M_{xy}\end{Bmatrix} \tag{5}$$

其中 $$D=\frac{Eh^3}{12(1-\mu^2)}$$

E为弹性模量，μ为泊松比，h为板的厚度。

2.2 以$[\psi_x\quad Q_x\quad M_x\quad w]$为对偶变量的对偶方程

选取四个对偶变量，组成向量$\{\bar{v}\}$

$$\{\bar{v}\}=[\psi_x\quad Q_x\quad M_x\quad w]^{\mathrm{T}} \tag{6}$$

其它变量——用$\{\bar{v}\}$及其对 y 的导数表示如下：

$$\kappa_y=-\frac{\partial^2 w}{\partial y^2} \tag{7a}$$

$$\kappa_{xy}=-2\frac{\partial \psi_x}{\partial y} \tag{7b}$$

$$\kappa_x=-\mu\kappa_y+\frac{M_x}{D}=\mu\frac{\partial^2 w}{\partial y^2}+\frac{M_x}{D}=-\frac{\partial \psi_x}{\partial x} \tag{7c}$$

$$M_y=\mu M_x+(1-\mu^2)D\kappa_y=\mu M_x-(1-\mu^2)D\frac{\partial^2 w}{\partial y^2} \tag{7d}$$

$$M_{xy}=\frac{1-\mu}{2}D\kappa_{xy}=-(1-\mu)D\frac{\partial \psi_x}{\partial y} \tag{7e}$$

$$\begin{aligned}Q_y&=\frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x}\\&=\left[\mu\frac{\partial M_x}{\partial y}-(1-\mu^2)D\frac{\partial^3 w}{\partial y^3}\right]-(1-\mu)D\frac{\partial}{\partial y}\left(\frac{\partial \psi_x}{\partial x}\right)\\&=\frac{\partial M_x}{\partial y}-(1-\mu)D\frac{\partial^3 w}{\partial y^3}\end{aligned} \tag{7f}$$

对偶微分方程——由式(7c)，式(3c)和(7f)，式(3a)和(7e)，(1a)可导出对偶微分方程如下：

$$\begin{cases}\dot{\psi}_x=-\dfrac{M_x}{D}-\mu\dfrac{\partial^2 w}{\partial y^2}\\[2mm]\dot{Q}_x=-\dfrac{\partial Q_y}{\partial y}-q=-\dfrac{\partial^2 M_x}{\partial y^2}+(1-\mu)D\dfrac{\partial^4 w}{\partial y^4}-q\\[2mm]\dot{M}_x=Q_x-\dfrac{\partial M_{xy}}{\partial y}=Q_x+(1-\mu)D\dfrac{\partial^2 \psi_x}{\partial y^2}\\[2mm]\dot{w}=\psi_x\end{cases} \tag{8}$$

式(8)可写成矩阵形式：

$$\{\dot{\bar{v}}\}=[\bar{H}]\{\bar{v}\}+\{h\} \tag{9}$$

其中

$$[\bar{H}]=\left[\begin{array}{cc:cc}0&0&-\dfrac{1}{D}&-\mu\dfrac{\partial^2}{\partial y^2}\\0&0&-\dfrac{\partial^2}{\partial y^2}&(1-\mu)D\dfrac{\partial^4}{\partial y^4}\\\hdashline(1-\mu)D\dfrac{\partial^2}{\partial y^2}&1&0&0\\1&0&0&0\end{array}\right],\ \{h\}=\begin{Bmatrix}0\\-q\\0\\0\end{Bmatrix} \tag{10}$$

2.3 以$[y_x \quad V_x \quad M_x \quad w]$为对偶变量的对偶方程

进行换元：

$$
\begin{aligned}
&V_x = Q_x + \frac{\partial M_{xy}}{\partial y} \\
&Q_x = V_x - \frac{\partial M_{xy}}{\partial y} = V_x + (1-m)D\frac{\partial^2 y_x}{\partial y^2}
\end{aligned} \tag{11}
$$

将对偶微分方程(8)改写成如下形式：

$$
\begin{cases}
\dot{y}_x = -\dfrac{M_x}{D} - m\dfrac{\partial^2 w}{\partial y^2} \\
\dot{V}_x = \dfrac{\partial Q_x}{\partial x} + \dfrac{\partial}{\partial y}\left(\dfrac{\partial M_{xy}}{\partial x}\right) \\
\quad = \left(-\dfrac{\partial Q_y}{\partial y} - q\right) + \dfrac{\partial}{\partial y}\left(Q_y - \dfrac{\partial M_y}{\partial y}\right) = -q - \dfrac{\partial^2 M_y}{\partial y^2} \\
\quad = -q - m\dfrac{\partial^2 M_x}{\partial y^2} + (1-m^2)D\dfrac{\partial^4 w}{\partial y^4} \\
\dot{M}_x = Q_x - \dfrac{\partial M_{xy}}{\partial y} = V_x - 2\dfrac{\partial M_{xy}}{\partial y} \\
\quad = V_x + 2(1-m)D\dfrac{\partial^2 y_x}{\partial y^2} \\
\dot{w} = y_x
\end{cases} \tag{12}
$$

上式可以写成矩阵形式

$$\{\dot{\boldsymbol{v}}\} = [\boldsymbol{H}]\{\boldsymbol{v}\} + \{\boldsymbol{h}\} \tag{13}$$

其中

$$
[\boldsymbol{H}] = \begin{bmatrix}
0 & 0 & -\dfrac{1}{D} & -m\dfrac{\partial^2}{\partial y^2} \\
0 & 0 & -m\dfrac{\partial^2}{\partial y^2} & (1-m^2)D\dfrac{\partial^4}{\partial y^4} \\
2(1-m)D\dfrac{\partial^2}{\partial y^2} & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}, \quad
\{\boldsymbol{h}\} = \begin{Bmatrix} 0 \\ -q \\ 0 \\ 0 \end{Bmatrix} \tag{14}
$$

3 薄板哈密顿变分原理泛函的推导特点

由文献[5]看到，厚板哈密顿变分原理的泛函P_H是以$w, y_x, y_y, M_x, M_{xy}, Q_x$为泛函变量。而厚板 Hellinger-Reissner 变分原理的泛函P_{HR}是以$w, y_x, y_y, M_x, M_y, M_{xy}, Q_x, Q_y$为泛函变量。因此泛函$P_H$可以从泛函$P_{HR}$中将 M_y，Q_y消去后即可导出。因此，对于厚板来说，采用消元法即可由P_{HR}导出P_H。

对于薄板问题，情况与厚板问题不同。薄板哈密顿泛函P_H是以w, y_x, M_x, V_x为泛函变量，而薄板 Hellinger-Reissner 泛函P_{HR}是以w, M_x, M_y, M_{xy}为泛函变量。因此由薄板P_{HR}到薄板P_H不仅要消去变量M_y和M_{xy}，而且还要增加变量y_x和V_x。因此对于薄板P_H来说，不能套用厚板P_H的推导方法，而需另辟蹊径。

由薄板P_{HR}（w, M_x, M_y, M_{xy}）推导薄板P_H(w, y_x, M_x, V_x)的步骤可分为两步，第 1 步是消元(消去M_y和M_{xy})，由P_{HR}(w, M_x, M_y, M_{xy})导出P(w, M_x)，用的是消元法；第 2 步是增元(增加y_x和V_x)，由P(w, M_x)导出P_H(w, y_x, M_x, V_x)，用的是换元乘子法。其推导过程可表示如下：

消元法

$\Pi_{HR}(w, M_x, M_y, M_{xy})$ $\Pi(w, M_x)$

换元乘子法

$\Pi_H(w, y_x, M_x, V_x)$

下面按此流程作详细论述，先用消元法导出P(w, M_x)，再用换元乘子法导出P_H(w, y_x, M_x, V_x)。

4 消元法——由薄板$P_{HR}(w, M_x, M_y, M_{xy})$导出$P(w, M_x)$

4.1 薄板泛函$P_{HR}(w, M_x, M_y, M_{xy})$及其自然条件

Hellinger-Reissner 泛函

$$P_{HR}(w, M_x, M_y, M_{xy}) = I_A + I_s \tag{15}$$

其中面积分I_A为

$$I_A = \iint_A [-B + P - qw]\mathrm{d}A \tag{16}$$

$$B = \frac{1}{2(1-m^2)D}[M_x^2 + M_y^2 - 2mM_xM_y + 2(1+m)M_{xy}^2] \tag{17}$$

$$P = -M_x\frac{\partial^2 w}{\partial x^2} - M_y\frac{\partial^2 w}{\partial y^2} - 2M_{xy}\frac{\partial^2 w}{\partial x \partial y} \tag{18}$$

边界积分I_s为

$$
\begin{aligned}
I_s = &\int_{S_2+S_3} M_n\frac{\partial w}{\partial n}\mathrm{d}S - \int_{S_3} V_n w\mathrm{d}S - \sum_{J_P} Pw \\
&+ \int_{S_1}(\frac{\partial w}{\partial n} - y_n)M_n\mathrm{d}S - \int_{S_1+S_2}(w-w)V_n\mathrm{d}S \\
&- \sum_{J_w}(w-w)(DM_{ns})
\end{aligned} \tag{19}
$$

S_1为固支边，S_2为简支边，S_3为自由边，J_P为无支撑角点，J_w为有支撑角点。

自然条件——包括下列方程和条件：

平衡微分方程：

$$\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+q=0 \qquad (20)$$

$M-w$**关系：**

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy}\end{Bmatrix}=D\begin{bmatrix}1 & m & 0\\ m & 1 & 0\\ 0 & 0 & \frac{1-m}{2}\end{bmatrix}\begin{Bmatrix}-\frac{\partial^2 w}{\partial x^2}\\ -\frac{\partial^2 w}{\partial y^2}\\ -2\frac{\partial^2 w}{\partial x\partial y}\end{Bmatrix}=-D\begin{Bmatrix}\frac{\partial^2 w}{\partial x^2}+m\frac{\partial^2 w}{\partial y^2}\\ m\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\\ (1-m)\frac{\partial^2 w}{\partial x\partial y}\end{Bmatrix} \qquad (21)$$

边界条件及角点条件：

$$\begin{aligned} & w=w,\ y_n=y_n && (\text{在}S_1\text{上})\\ & w=w,\ M_n=M_n && (\text{在}S_2\text{上})\\ & M_n=M_n,\ V_n=V_n && (\text{在}S_3\text{上})\\ & w=w && (\text{在}J_w\text{上})\\ & DM_{ns}=P && (\text{在}J_P\text{上})\end{aligned} \qquad (22)$$

这里l**，**m**为边界向外法线的方向余弦，**

$$\begin{aligned} y_n &= \frac{\partial w}{\partial n}=l\frac{\partial w}{\partial x}+m\frac{\partial w}{\partial y}\\ M_n &= l^2M_x+m^2M_y+2lmM_{xy}\\ M_{ns} &= (-M_x+M_y)lm+(l^2-m^2)M_{xy}\\ V_n &= Q_n+\frac{\partial M_{ns}}{\partial s}\\ Q_n &= lQ_x+mQ_y\\ &= l\left(\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}\right)+m\left(\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}\right)\\ \frac{\partial}{\partial s} &= -m\frac{\partial}{\partial x}+l\frac{\partial}{\partial y}\end{aligned} \qquad (23)$$

4.2 **消元法由**Π_{HR}**导出**$\Pi(w, M_x)$

在Π_{HR}**中消去泛函变量**M_y**和**M_{xy}**，将相关的两个自然条件：**

$$\begin{aligned} M_y &= -D\left(m\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\\ M_{xy} &= -(1-m)D\frac{\partial^2 w}{\partial x\partial y}\end{aligned} \qquad (24)$$

代入泛函Π_{HR}**，得到新泛函**

$$P(w,M_x)=I_A^*+I_s^* \qquad (25)$$

上式中面积分 I_A^* **为**

$$I_A^*=\iint_A[-B^*+P^*-qw]\mathrm{d}A \qquad (26)$$

$$\begin{aligned} B^* = \frac{D}{2(1-m^2)}[\frac{M_x^2}{D^2}+(\frac{\partial^2 w}{\partial y^2}+m\frac{\partial^2 w}{\partial x^2})^2 &+\frac{2m}{D}M_x(\frac{\partial^2 w}{\partial y^2}+mm\frac{\partial^2 w}{\partial x^2})\\ &+2(1+m)(1-m)^2(\frac{\partial^2 w}{\partial x\partial y})^2]\end{aligned} \qquad (27)$$

$$\begin{aligned} P^* = -M_x\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}D(\frac{\partial^2 w}{\partial y^2}+m\frac{\partial^2 w}{\partial x^2}) +2(1-m)D(\frac{\partial^2 w}{\partial x\partial y})^2\end{aligned} \qquad (28)$$

整理后得

$$I_A^*=\iint_A[A(w)-\frac{D}{2(1-m^2)}(\frac{M_x}{D}+\frac{\partial^2 w}{\partial x^2}+m\frac{\partial^2 w}{\partial y^2})^2-qw]\mathrm{d}A \qquad (29)$$

其中$A(w)$**为薄板的应变能密度：**

$$A(w)=\frac{D}{2}\{(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})^2+2(1-m)[(\frac{\partial^2 w}{\partial x\partial y})^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}]\} \qquad (30)$$

式(25)**中线积分** I_s^* **为**

$$\begin{aligned} I_s^* = &\int_{S_2+S_3}M_n\frac{\partial w}{\partial n}\mathrm{d}S-\int_{S_3}V_n w\mathrm{d}S-\sum_{J_P}Pw\\ &+\int_{S_1}(\frac{\partial w}{\partial n}-y_n)\hat{M}_n\mathrm{d}S-\int_{S_1+S_2}(w-w)\hat{V}_n\mathrm{d}S\\ &-\sum_{J_w}(w-w)(DM_{ns})\end{aligned} \qquad (31)$$

其中

$$\begin{aligned} \hat{M}_n &= l^2M_x-m^2D\left(m\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-2lm(1-m)D\frac{\partial^2 w}{\partial x\partial y}\\ \hat{V}_n &= \hat{Q}_n+\frac{\partial \hat{M}_{ns}}{\partial s}\\ \hat{Q}_n &= l\left(\frac{\partial M_x}{\partial x}\right)-D(1-m)\left(l\frac{\partial}{\partial y}+m\frac{\partial}{\partial x}\right)\frac{\partial^2 w}{\partial x\partial y}\\ &\quad -Dm\frac{\partial}{\partial y}\left(m\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\\ \hat{M}_{ns} &= l^2M_x-m^2D\left(m\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-2lm(1-m)D\frac{\partial^2 w}{\partial x\partial y}\end{aligned} \qquad (32)$$

5 换元乘子法——由薄板$P(w, M_x)$ 导出$P_H(w, M_x, y_x, V_x)$

为了将y_x引入到新泛函，可将关系式

$$y_x = \frac{\partial w}{\partial x} \tag{33}$$

代入泛函$\Pi(w, M_x)$，得出换元后的新泛函

$$P(w,M_x,y_x) = \iint_A [A(w,y_x) - \frac{D}{2(1-m^2)}(\frac{M_x}{D} + \frac{\partial y_x}{\partial x} + m\frac{\partial^2 w}{\partial y^2})^2 - qw]\mathrm{d}A \tag{34}$$

$+$线积分项

其中

$$A(w,y_x) = \frac{D}{2}\left\{\left(\frac{\partial y_x}{\partial x} + \frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1-m)\left[\left(\frac{\partial y_x}{\partial y}\right)^2 - \frac{\partial y_x}{\partial x}\frac{\partial^2 w}{\partial y^2}\right]\right\} \tag{35}$$

这里式(33)是泛函$\Pi(w, M_x, y_x)$的强制条件。为了将式(33)由强制条件转变为自然条件，采用乘子法得出新泛函

$$P(w,M_x,y_x,l) = P(w,M_x,y_x) + \iint_A l(\frac{\partial w}{\partial x} - y_x)\mathrm{d}A \tag{36}$$

求其变分得

$$\begin{aligned} dP(w,M_x,y_x,l) = \iint_A \{ & dl(\frac{\partial w}{\partial x} - y_x) \\ & - dM_x\frac{1}{1-m^2}(\frac{M_x}{D} + \frac{\partial y_x}{\partial x} + m\frac{\partial^2 w}{\partial y^2}) \\ & - dy_x D[(\frac{\partial^2 y_x}{\partial x^2} + 2(1-m)\frac{\partial^2 y_x}{\partial y^2} + m\frac{\partial^3 w}{\partial x \partial y^2}) \\ & - \frac{1}{1-m^2}\frac{\partial}{\partial x}(\frac{M_x}{D} + \frac{\partial y_x}{\partial x} + m\frac{\partial^2 w}{\partial y^2})] \\ & + dwD[(m\frac{\partial^3 y_x}{\partial x \partial y^2} + \frac{\partial^4 w}{\partial y^4}) \\ & - \frac{m}{1-m^2}\frac{\partial^2}{\partial y^2}(\frac{M_x}{D} + \frac{\partial y_x}{\partial x} + m\frac{\partial^2 w}{\partial y^2}) - \frac{q}{D}]\}\mathrm{d}A \end{aligned}$$

$+$线积分项 (37)

因此，$d\Pi(w, M_x, y_x, l)=0$的欧拉方程为

$$\frac{\partial w}{\partial x} - y_x = 0 \tag{38a}$$

$$\frac{M_x}{D} + \frac{\partial y_x}{\partial x} + m\frac{\partial^2 w}{\partial y^2} = 0 \tag{38b}$$

$$l + D[\frac{\partial}{\partial x}(\frac{\partial y_x}{\partial x} + m\frac{\partial^2 w}{\partial y^2}) + 2(1-m)\frac{\partial^2 y_x}{\partial y^2}] = l - \frac{\partial M_x}{\partial x} + 2(1-m)D\frac{\partial^2 y_x}{\partial y^2} = 0 \tag{38c}$$

$$-\frac{\partial l}{\partial x} + D\frac{\partial^2}{\partial y^2}(m\frac{\partial y_x}{\partial x} + \frac{\partial^2 w}{\partial y^2}) - q = -\frac{\partial l}{\partial x} - m\frac{\partial^2 M_x}{\partial y^2} + (1-m^2)D\frac{\partial^4 w}{\partial y^4} - q = 0 \tag{38d}$$

由(38c)得知

$$l = \frac{\partial M_x}{\partial x} - 2(1-m)D\frac{\partial^2 y_x}{\partial y^2} = Q_x + \frac{\partial M_{xy}}{\partial y} = V_x \tag{39}$$

于是可知乘子l可换名为V_x。乘子换名后的泛函为

$$P_H(w,M_x,y_x,V_x) = \iint_A [A(w,y_x) - \frac{D}{2(1-m^2)}(\frac{M_x}{D} + \frac{\partial y_x}{\partial x} + m\frac{\partial^2 w}{\partial y^2})^2 - qw + V_x(\frac{\partial w}{\partial x} - y_x)]\mathrm{d}A + \text{线积分项} \tag{40}$$

此即薄板哈密顿变分原理的泛函。

乘子换名后，欧拉方程式(38)即为哈密顿对偶方程(12)。

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