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一维分数导数粘弹性饱和多孔介质层的稳态响应

刘林超, 闫启方

(信阳师范学院土木工程学院, 河南, 信阳 464000)

摘 要: 固相骨架的应力-应变关系利用分数导数粘弹性 Kelvin 模型来描述, 在流相和固相微观不可压以及小变形的假定下建立了分数导数粘弹性饱和多孔介质层一维稳态响应的数学模型和运动控制方程, 求得了分数导数粘弹性饱和多孔介质层一维稳态响应的固相位移和液相位移。通过数值算例分析了分数导数的阶数对稳态响应的影响。研究表明: 固相位移和液相位移随频率的增大逐渐趋于零, 在低频时, 分数导数的阶数越大固相位移和液相位移越大。

关键词: 分数导数; 粘弹性; 动力响应; 多孔介质; 不可压

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STEADY STATE RESPONSE OF ONE-DIMENSION LIQUID-SATURATED POROUS MEDIUM USING FRACTIONAL DERIVATIVE VISCOELASTIC MODEL

LIU Lin-chao, YAN Qi-fang

(School of Civil Engineering, Xinyang Normal University, Xinyang, Henan 464000, China)

Abstract: The stress-strain relationship of a solid skeleton is described by a fractional derivative Kelvin viscoelastic model, and the mathematic model and equations of motion of the steady state response of one-dimension liquid-saturated porous medium are established, in which the saturated porous material is modeled as a two phase system composed of an incompressible solid phase and an incompressible fluid phase, and the displacements both of the solid phase and fluid phase in one dimension liquid saturated porous medium described by fractional derivative viscoelastic model are obtained. The influence of fractional order on the steady state response is analyzed by a numerical example. The result indicates that the displacements both of solid phase and fluid phase will decrease to zero with the increase of frequency, and the displacements both of solid phase and fluid phase increase with the increase of fractional order at lower frequencies.

Key words: fractional derivative; viscoelasticity; dynamic response; porous medium; incompressible

科学技术的发展, 使得流体饱和多孔介质理论在土力学、水文学等学科中发挥着重要的作用, 同时也是许多新兴学科和应用技术发展的关键^[1]。具有固相粘性的多孔介质存在于众多的合成和天然材料中, 对其研究具有重要的应用价值。目前的研究大多是假设固相骨架为弹性或弹塑性材料, 应用 Biot 理论、杂交混合物理论和多孔介质理论对多孔

介质的静力、动力耦合响应进行理论分析或数值模拟。Edelman 等^[2]指出 Biot 理论模型中的耦合惯性质量为非客观量, 且物相间相互作用力的本构假定不满足物质客观性公理, 而杂交混合物理论则需要繁杂的公式推导, 更为合理的理论是多孔介质理论。基于多孔介质理论, Liu 和 De Boer^[3-4]对不可压流体饱和多孔介质半空间中平面波的传播和反射等进

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通讯作者: 刘林超(1979—), 男, 河南漯河人, 讲师, 博士生, 从事固体力学、粘弹性理论等研究(E-mail: llc109@126.com)。

作者简介: 闫启方(1979—), 女, 河南驻马店人, 讲师, 硕士生, 从事粘弹性理论等研究(E-mail: yqf815@126.com)。

行了研究。刘占芳等^[5-6]对饱和多孔介质中 Rayleigh 波和 Love 波、非均匀平面波的传播特征和动力学行为进行了研究。杨晓等^[7]基于多孔介质理论和 Crump 数值反演方法对流体饱和和不可压粘弹性多孔介质层的一维动力响应进行了研究。由于分数导数粘弹性模型具有模型简单, 确定模型所需要的实验参数少, 能在较宽的频率范围内描述材料的力学行为开始引起众多学者的关注^[8-10]。本文将在分数导数粘弹性理论和多孔介质理论的基础上, 研究一维分数导数粘弹性饱和介质层的稳态响应问题, 并分析分数导数的阶数对稳态响应的影响规律。

1 数学模型及控制方程

考察图 1 所示的半无限流体饱和粘弹性多孔介质, 它由不相溶的微观不可压流相和微观不可压固相组成, 其表面作用有均布简谐荷载 $\bar{q}_0(t) = \bar{q}e^{i\omega t}$ 。忽略固相骨架和孔隙流体间的质量和能量交换以及各自的体积力, 对于沿 x 方向的一维耦合动力学响应, 其场控制方程可由多孔介质理论表示为^[3,7]:

$$\frac{\partial \bar{\sigma}^{SE}}{\partial x} - \frac{\partial \bar{p}}{\partial x} - \rho^S \frac{\partial^2 \bar{u}^S}{\partial t^2} + s_v \left(\frac{\partial \bar{w}^F}{\partial t} - \frac{\partial \bar{u}^S}{\partial t} \right) = 0 \quad (1)$$

$$-n^F \frac{\partial \bar{p}}{\partial x} - \rho^F \frac{\partial^2 \bar{w}^F}{\partial t^2} - s_v \left(\frac{\partial \bar{w}^F}{\partial t} - \frac{\partial \bar{u}^S}{\partial t} \right) = 0 \quad (2)$$

$$\frac{\partial}{\partial x} \left(n^S \frac{\partial \bar{u}^S}{\partial t} + n^F \frac{\partial \bar{w}^F}{\partial t} \right) = 0 \quad (3)$$

式中: n^S 、 n^F 分别为固相土骨架和液相的体积分数; $\bar{u}^S$ 、 $\bar{w}^F$ 分别为固相土骨架和液相的位移; $\bar{p}$ 为孔隙水压力; $\bar{\sigma}^{SE}$ 为固相土骨架沿 x 方向的有效应力; ρ^S 、 ρ^F 分别为固相土骨架和液相的宏观质量密度; $s_v = \frac{n^F \gamma^{FR}}{k^F}$ 为液相和固相的耦合系数; γ^{FR} 为孔隙水的比重; k^F 为 Darcy 渗透系数。

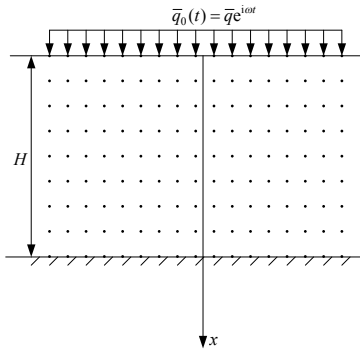


图 1 分数导数粘弹性饱和多孔介质层

Fig.1 Porous medium layer described by FDV model

假设流体饱和多孔介质层的底部为刚性和非排水, 上表面为理想排水且作用有均布简谐荷载 $\bar{q}_0(t) = \bar{q}e^{i\omega t}$, 则相应的边界条件为:

$$\begin{cases} \bar{\sigma}^{SE}(x, t) = -\bar{q}_0(t), \bar{p}(x, t) = 0, & x = 0 \\ \bar{u}^S(x, t) = 0, \bar{w}^F(x, t) = 0, & x = H \end{cases} \quad (4)$$

设流体饱和多孔介质初始未变形, 则有:

$$\bar{u}^S(x, 0) = 0, \frac{\partial \bar{u}^S(x, 0)}{\partial t} = 0, \bar{w}^F(x, 0) = 0 \quad (5)$$

固相骨架采用分数导数 Kelvin 粘弹性模型来描述其应力-应变关系^[11]:

$$\bar{\sigma}^{SE}(x, t) = G_0 [\bar{\varepsilon}^S(x, t) + \eta^\alpha D^\alpha \bar{\varepsilon}^S(x, t)] \quad (6)$$

其中: G_0 、 η 为材料参数; D^α 是 Riemann-Liouville 分数微分算子, $0 < \alpha < 1$, 其定义为^[12]:

$$D^\alpha [x(t)] = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{x(\tau)}{(t-\tau)^\alpha} d\tau \quad (7)$$

其中: $\Gamma(\cdot)$ 为 Gamma 函数, $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$, $\text{Re}(z) > 0$ 。

固相骨架的应变-位移关系为:

$$\bar{\varepsilon}^S(x, t) = \frac{\partial \bar{u}^S(x, t)}{\partial x} \quad (8)$$

由式(6)和式(8)可得固相骨架应力和位移的关系为:

$$\bar{\sigma}^{SE}(x, t) = G_0 \left(1 + \eta^\alpha \frac{d^\alpha}{dt^\alpha} \right) \frac{\partial \bar{u}^S(x, t)}{\partial x} \quad (9)$$

由式(1)~式(3)、式(9)可得一维情况下分数导数粘弹性饱和多孔介质的运动控制方程为:

$$G_0 \left(1 + \eta^\alpha \frac{d^\alpha}{dt^\alpha} \right) \frac{\partial^2 \bar{u}^S(x, t)}{\partial x^2} - \frac{\partial \bar{p}}{\partial x} - \rho^S \frac{\partial^2 \bar{u}^S}{\partial t^2} + s_v \left(\frac{\partial \bar{w}^F}{\partial t} - \frac{\partial \bar{u}^S}{\partial t} \right) = 0 \quad (10)$$

$$-n^F \frac{\partial \bar{p}}{\partial x} - \rho^F \frac{\partial^2 \bar{w}^F}{\partial t^2} - s_v \left(\frac{\partial \bar{w}^F}{\partial t} - \frac{\partial \bar{u}^S}{\partial t} \right) = 0 \quad (11)$$

$$\frac{\partial}{\partial x} \left(n^S \frac{\partial \bar{u}^S}{\partial t} + n^F \frac{\partial \bar{w}^F}{\partial t} \right) = 0 \quad (12)$$

2 一维分数导数粘弹性饱和多孔介质稳态响应的解

由于问题为稳态响应, 所以各参量具有 $\bar{f} = \tilde{f}e^{i\omega t}$ 的形式, 将其代入式(10)~式(12)并在以后的计算中忽略 $e^{i\omega t}$ 项, 同时考虑初始条件式(5), 可得:

$$G_0 [1 + \eta^\alpha (i\omega)^\alpha] \frac{\partial^2 \tilde{u}^S}{\partial x^2} - \frac{\partial \tilde{p}}{\partial x} +$$

$$\rho^S \omega^2 u^S + i\omega S_v (\tilde{w}^F - \tilde{u}^S) = 0 \quad (13)$$

$$-n^F \frac{\partial \tilde{p}}{\partial x} + \rho^F \omega^2 \tilde{w}^F - i\omega S_v (\tilde{w}^F - \tilde{u}^S) = 0 \quad (14)$$

$$\frac{\partial}{\partial x} (n^S \tilde{u}^S + n^F \tilde{w}^F) = 0 \quad (15)$$

$$\text{令 } \bar{x} = \frac{x}{H}, \bar{\omega} = \frac{H\omega}{v_s}, v_s = \sqrt{G_0 / \rho^S}, u^S = \tilde{u}^S / H,$$

$$w^F = \tilde{w}^F / H, S_v = \frac{H S_v}{v_s \rho^S}, \bar{\rho} = \frac{\rho^L}{\rho^S}, \kappa = \frac{\eta v_s}{H},$$

$$p = \frac{\tilde{p}}{G_0}, \text{代入式(13)~式(15)进行无量纲化运算可得:}$$

$$[1 + (i\kappa\bar{\omega})^\alpha] \frac{\partial^2 u^S}{\partial \bar{x}^2} - \frac{\partial p}{\partial \bar{x}} + \bar{\omega}^2 u^S + i\bar{\omega} S_v (w^F - u^S) = 0 \quad (16)$$

$$-n^F \frac{\partial p}{\partial \bar{x}} + \bar{\rho} \bar{\omega}^2 w^F - i\bar{\omega} S_v (w^F - u^S) = 0 \quad (17)$$

$$\frac{\partial}{\partial \bar{x}} (n^S u^S + n^F w^F) = 0 \quad (18)$$

利用矩阵运算, 将式(16)~式(18)改写为:

$$\mathbf{A}(\bar{\omega}) \frac{\partial^2}{\partial \bar{x}^2} \mathbf{Y}(\bar{x}, \bar{\omega}) + \mathbf{B}(\bar{\omega}) \frac{\partial}{\partial \bar{x}} \mathbf{Y}(\bar{x}, \bar{\omega}) + \mathbf{C}(\bar{\omega}) \mathbf{Y}(\bar{x}, \bar{\omega}) = 0 \quad (19)$$

其中:

$$\mathbf{A}(\bar{\omega}) = \begin{bmatrix} 1 + (i\kappa\bar{\omega})^\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \mathbf{B}(\bar{\omega}) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -n^F \\ n^S & n^F & 0 \end{bmatrix};$$

$$\mathbf{C}(\bar{\omega}) = \begin{bmatrix} \bar{\omega}^2 - i\bar{\omega} S_v & i\bar{\omega} S_v & 0 \\ i\bar{\omega} S_v & \bar{\rho} \bar{\omega}^2 - i\bar{\omega} S_v & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

$$\mathbf{Y}(\bar{x}, \bar{\omega}) = \begin{bmatrix} u^S \\ w^F \\ p \end{bmatrix} \quad (20)$$

设问题的解的形式为:

$$\mathbf{Y}(\bar{x}, \bar{\omega}) = \mathbf{Y}_0(\bar{\omega}) e^{\beta(\bar{\omega})\bar{x}} \quad (21)$$

将式(21)代入式(19)可得:

$$(\beta^2 \mathbf{A} + \beta \mathbf{B} + \mathbf{C}) \mathbf{Y}_0 e^{\beta\bar{x}} = 0 \quad (22)$$

求解式(22), 由微分方程理论并考虑边界条件, 可得式(19)的通解为:

$$\mathbf{Y}(\bar{x}, \bar{\omega}) = C_1 \mathbf{Y}_1 e^{\chi(\bar{\omega})\bar{x}} + C_2 \mathbf{Y}_2 e^{-\chi(\bar{\omega})\bar{x}} + C_3 \mathbf{Y}_3 \quad (23)$$

其中:

$$\chi^2 = \frac{n^S (\bar{\rho} \bar{\omega}^2 - i\bar{\omega} S_v) - \bar{\omega}^2 n^{F2}}{n^{F2} [1 + (i\kappa\bar{\omega})^\alpha]};$$

$$\mathbf{Y}_1 = \mathbf{Y}_2 = \begin{bmatrix} -\frac{n^F}{n^S} \\ 1 \\ \frac{n^S \bar{\rho} \bar{\omega}^2 - i\bar{\omega} S_v}{n^S n^F \chi} \end{bmatrix}; \mathbf{Y}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (24)$$

待定系数 C_1 、 C_2 、 C_3 可由边界条件确定。对边界条件式(4)进行 Fourier 变换和无量纲化运算, 可得:

$$\begin{cases} \sigma^{SE}(\bar{x}, \bar{\omega}) = -q, & p(\bar{x}, \bar{\omega}) = 0, & \bar{x} = 0 \\ u^S(\bar{x}, \bar{\omega}) = 0, & w^F(\bar{x}, \bar{\omega}) = 0, & \bar{x} = 1.0 \end{cases} \quad (25)$$

其中: $\sigma^{SE} = \bar{\sigma}^{SE} / G_0$; $q = \tilde{q} / G_0$ 。由式(23)、式(24)得:

$$C_1 = \frac{q}{[1 + (i\kappa\bar{\omega})^\alpha] \frac{n^F}{n^S} \chi(\bar{\omega}) (1 + e^{2\chi(\bar{\omega})})},$$

$$C_2 = -e^{2\chi(\bar{\omega})} C_1,$$

$$C_3 = -\frac{n^S \bar{\rho} \bar{\omega}^2 - i\bar{\omega} S_v}{n^S n^F \chi} \frac{q(1 - e^{2\chi(\bar{\omega})})}{[1 + (i\kappa\bar{\omega})^\alpha] \frac{n^F}{n^S} \chi(\bar{\omega})}。$$

进而可得一维分数导数粘弹性饱和介质层的稳态响应为:

$$u^S = -C_1 \frac{n^F}{n^S} e^{\chi(\bar{\omega})\bar{x}} - C_2 \frac{n^F}{n^S} e^{-\chi(\bar{\omega})\bar{x}} \quad (26)$$

$$w^F = -C_1 e^{\chi(\bar{\omega})\bar{x}} - C_2 e^{-\chi(\bar{\omega})\bar{x}} \quad (27)$$

$$p = C_1 \frac{n^S \bar{\rho} \bar{\omega}^2 - i\bar{\omega} S_v}{n^S n^F \chi} e^{\chi(\bar{\omega})\bar{x}} + C_2 \frac{n^S \bar{\rho} \bar{\omega}^2 - i\bar{\omega} S_v}{n^S n^F \chi} e^{-\chi(\bar{\omega})\bar{x}} + C_3 \quad (28)$$

3 数值算例与讨论

这里以图 1 所示的一维饱和粘弹性多孔介质层为例, 以数值算例的形式重点考察分数导数的阶数 α 对一维分数导数粘弹性饱和介质层稳态响应的影响。图 2 和图 3 给出了分数导数的阶数 $\alpha=0.2$ 、 $\alpha=0.4$ 、 $\alpha=0.6$ 、 $\alpha=0.8$ 、 $\alpha=1.0$ 时, 粘弹性多孔介质层表面无量纲固相位移和液相位移随无量纲频率的变化曲线, 相关参数取值为: $S_v=0.05$, $\kappa=0.8$, $n^S=0.67$, $n^F=0.33$ 。从固相位移和液相位移随频率的变化曲线可以看出, 曲线随频率的增大逐渐趋于零, 这是由于频率较大时体系还没来得及发生位移就已经向相反的方向变形, 且在相同参数情况下, 液相位移和

固相位移随频率的变化规律较为接近,但液相位移比固相位移要大。从图2和图3可以看出弹性多孔介质层的固相位移和液相位移较粘弹性的解要大。在低频时分数导数粘弹性饱和多孔介质层受分数导数的阶数 α 的影响较大,而高频时由于位移趋于零所以影响不大,且在低频时分数导数的阶数 α 越大,固相位移和液相位移则越大,由于分数导数阶数的确定可以根据蠕变试验的数据进行拟合得到, α 反映了粘弹性饱和多孔介质层的力学特性,对于不同的粘弹性饱和多孔介质层,我们可以通过 α 等模型参数来体现,可见分数导数粘弹性模型可以在较大的范围内描述粘弹性饱和土多孔介质层的力学特性。另外,分数导数粘弹性饱和多孔介质的固相位移和液相位移可以退化到经典粘弹性($\alpha=1$)和弹性的情况。

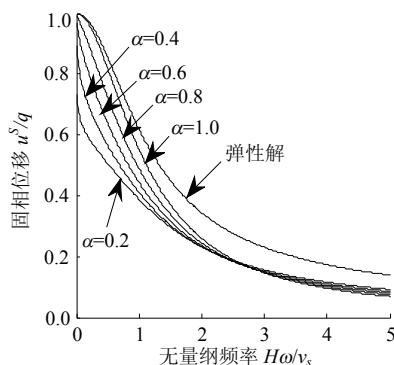


图2 分数导数粘弹性多孔介质的固相位移

Fig.2 Displacement of the solid skeleton for porous medium described by fractional derivative viscoelastic model

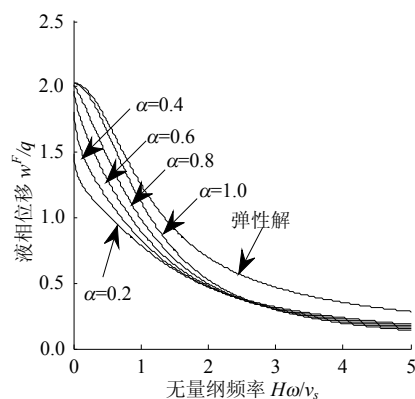


图3 分数导数粘弹性多孔介质的液相位移

Fig.3 Displacement of the fluid for porous medium described by fractional derivative viscoelastic model

4 结论

基于粘弹性理论、分数导数理论和多孔介质理论,研究了分数导数不可压流体粘弹性饱和多孔介

质层的一维稳态响应问题,给出了固相位移、液相位移和孔水压力的数学表达式,并重点讨论了分数导数的阶数对饱和土多孔介质层固相和液相位移的影响。研究发现分数导数的阶数对一维分数导数粘弹性饱和和介质层的稳态响应有影响,且可以退化为经典粘弹性和弹性解的情况,说明了本文分析方法的正确性;在低频和高频时,分数导数的阶数对粘弹性饱和和多孔介质层的影响不同;分数导数粘弹性模型可以在较大的范围内描述粘弹性饱和多孔介质层的力学行为。本文针对一维分数导数粘弹性多孔介质层的理论分析对建筑物物理中的湿度、热的迁移规律、生物力学中皮肤、软组织和关节软骨的力学性能、地热储层的合理开发和利用问题中的流固耦合问题等方面的研究有着重要的学术价值和广泛的应用前景。

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